LILLIE R. ALBERT

OUTSIDE-IN – INSIDE-OUT: SEVENTH-GRADE STUDENTS' MATHEMATICAL THOUGHT PROCESSES

ABSTRACT. Building on the research of Vygotsky regarding the role of social interaction and the zone of proximal development (ZPD) in learning and development, this paper explored the relation between students' oral thought processes and written thought processes. It is argued that the practice of writing provides a context for a new learning zone: the 'zone of proximal practice' (ZPP). In this new zone, students independently organize their thinking about mathematical concepts and ideas. An interpretative case study of seven middle grade students is presented to support this contention. The case study describes the strategies and procedures students employed while solving mathematical problems and documents students' oral and written thought processes through interview protocols and writing samples. The position that students' mathematical understanding is further developed through writing as a communicative tool, while taking advantage of mediated social practices, is discussed to make clear the rationale for introducing a new learning zone.

1. INTRODUCTION

When students synthesize information, they organize more than one idea into a single concept. This process involves working with individual pieces of information and rearranging them in such a way as to construct a new pattern or structure. A way to accomplish this is through language in the form of conversations or writing. The act of writing serves as a mode for students to reflect on their thinking. This method of communication allows them to convey ideas, feelings, and experiences that can lead to the development of higher cognitive functions, including critical thinking, sound reasoning, and problem-solving. These practices are closely aligned with the major features of Vygotsky's (1978) construct, 'the zone of proximal development'.

Vygotsky (1978) contended that the development of higher cognitive functions is launched within the zone of proximal development (ZPD). Learning within the ZPD occurs when students are involved with tasks or problems that go beyond their immediate individual capabilities in which teachers (or other adults) assist their performance, or in collaboration with more knowledgeable peers. Central to this process of learning is the role of

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social interaction in the development of higher cognitive functions that occurs on two planes. On the first plane, students work with their peers in collaborative situations or complete activities facilitated by the teacher. Vygotsky, Wertsch (1979) and other investigators (e.g., Newman and Holzman, 1993; Rogoff, 1990; Rogoff and Wertsch, 1984; Tudge, 1990) refer to this level as the interpsychological plane (the social level). Social interactions lead to independent thinking, reasoning, or problem solving within the intrapsychological plane, the second plane. As Vygotsky proposed, '[a]ll the higher functions originate as actual relations between [people]' (1978, p. 57). The major thesis, as in all Vygotsky's work, rests on the assumption that development cannot be separated from social contexts or from language, oral or written.

What, however, exists beyond the zone of proximal development? How can we describe where individual students perform once they have achieved self-regulation? If writing establishes the environment in which the zone of proximal development is embedded, is it possible to conceptualize the learner's liberation from the assisting scaffold, or aid, within the ZPD, in which the learner moves from working with assistance within a social context, to working unassisted? How can we define this transition to self-regulatory behavior? As students face new challenges, new problems, new learning processes, they can be characterized as functioning progressively within several ZPDs – each of which require new scaffolding. At some point, however, the character of the learner's scaffolding ability transforms; the learner can do something independently today that he could only do with assistance yesterday. This transformation may be manifested in terms of concrete, observable behavior, or may be metacognitive in nature. The end result of this type of metacognitive change can be conceptualized in terms of a zone in which the learner directs the literacy process and applies this knowledge to reorganizing future experiences or activities. A new zone of self-regulation, or self-assisted practice emerges, where a type of 'self-scaffolding' occurs: the zone of proximal practice (ZPP).

This paper reports on a study that explored the relation between oral and written thought processes of seven middle grade students. In particular, it examined problem solving procedures and strategies used by students to articulate their understanding of fundamental mathematical concepts. It is grounded in Vygotsky's (1978) major construct concerning the role of social interactions and the ZPD to introduce the notion that the ZPD supports students' oral and written ideas about problem-solving. In the newly conceived 'zone of proximal practice' (ZPP) introduced in this body of work, students independently practice and organize their thinking about mathematical concepts and ideas through writing and graphic representations.

Originating in social thought, writing functions as a tool for expressing individual meaning and thinking, enabling the learner's self-scaffolding and providing the bridge from the ZPD to the ZPP.

As a means of thinking about the way writing enhances self-assisted practice, and becomes the product of self-regulatory processes, a brief summary of the theoretical assumptions regarding the development of written language advocated by Vygotsky (1986), Emig (1983), and others (Burkhalter, 1995; DiCamilla and Anton, 1997; Everson, 1991; Freedman, 1994) is discussed. It has been suggested that the development of cognitive functions (e.g., evaluating the reasonableness of a solution) is supported by the practice of writing (Elbow, 1981; Graves, 1983; Hoel, 1997; John-Steiner, 1985; Moffett, 1981; Smith, 1988). This allows us to think broadly and systematically about how metacognitive thought and oral conversations have a natural meeting point in writing that creates a process in which meaningful practice takes place. The work of Bruner (1971) and Luria and Yudovich (1971) is drawn upon to complement and expand the construct of the ZPD in order to introduce the notion of a new zone, the zone of proximal practice (ZPP). The assertion is made that writing – only one of many language tools (e.g., oral language, gestures, and semiotic systems) – is a device for mediating cognitive development, moving the learner through the zone of proximal development to the zone of proximal practice. To explore how collaborative groups served as a catalyst for reflective writing, the following section analyzes the interview dialogue of students working in a collaborative context. It is within this arena that individual students' knowledge came together while engaging in a socially constructed context. The final discussion revisits the theoretical framework of the study to assess the research findings by reconsidering data interpretation to make clear the rationale for introducing a *new 'learning zone'*.

2. The Development of Writing

The studies of concept formation in educational settings conducted by Lev Vygotsky in the 1930s proposed that the development of writing does not represent the developmental history of speaking. Vygotsky argued that writing is a separate linguistic function from oral language both in structure and mode of functioning. Thus, writing is viewed as something that one does alone, without the need for an audience. Vygotsky drew the analogy that 'just as learning an algebraic formula does not repeat arithmetic skills, the development of writing does not repeat the development of oral speech' (1986, p. 181).

Vygotsky (1986) maintained that the purposes for writing are more theoretical than the motives for speaking. The act of writing, stated Vygotsky requires analytical behavior that is a more conscious and deliberate act than speaking. Vygotsky observed that writing enhances the cognitive actions undertaken by children and that inner speech plays a significant role in the writing process. In particular, when students communicate in writing, they are relying on the formal meaning of words. Therefore, they use a greater number of words than they would use to represent the same ideas orally. Articulating one's thought processes in writing demands a greater number of words to express understanding of concepts or ideas (Vygotsky, 1986).

Emig (1983), an educational theorist, expanded the notion that writing is uniquely, logically, and theoretically different from other language processes such as listening, reading, and talking. Emig contended that '[b]ecause writing is often our representation of the word made visible, embodying both process and product, writing is more readily a form and source of learning than talking' (p. 125). Emig (1983) further explained that the process and product aspects of writing correspond to learning strategies. This assumption is related to earlier research work of Bruner, Oliver, and Greenfield (1966) and Luria and Yudovich (1971) who viewed writing as a learning strategy. They embraced the view that higher cognitive functions, such as analysis and synthesis, seem to develop most fully with the support of oral language, but more specifically with written language. Writing can be a powerful learning strategy because '[w]ritten speech ... assumes a slower, repeated mediating process of analysis and synthesis, which makes it possible not only to develop the required thought, but even to revert to its earlier stages, transforming the sequential chain of connections in a simultaneous, self-reviewing structure. Written speech represents a new and powerful instrument of thought' (Luria and Yudovich, 1971, p. 118). For example, when students use the approach described by Luria and Yudovich to solve problems, they can use their writing to reconstruct or reorganize the information given into several parts by answering the questions: 'What do I know about this problem?' 'What do I need to know to solve this problem?' 'What strategy do I need to solve this problem?'

Vygotsky (1986) claimed that written speech provides students with opportunities, such as social conversations with peers, through which thought is organized outside direct and informal contexts. Consequently, the social context of learning helps transform written speech because written language provides opportunities to use oral language out of a social context. The student is able to go from an interplane of learning to an intraplane of learning, from assisted practices to unassisted practices. In this respect,

'any written language used out of a concrete context should produce some cognitive results' (Bruner, 1971, p. 311).

Over the last two decades, writing research has come to recognize the contribution of collaborative discourse in individual thought processes (Everson, 1991). The role of inner speech on composing (Moffett, 1981) and the internal dialogue of our writer's mind (Elbow, 1981), illustrate the cyclical nature of social learning and the internalization (Vygotsky, 1978) of thought processes. Language learning argued Smith (1988) is a natural and social phenomena, suggesting that students learn more in a sharing and collaborative environment with peers and mentors. Building on this argument, Burkhalter (1995) contends that peer and mentor assistance aids individuals' oral argumentation skills and the ability to judge the strength of arguments. John-Steiner (1985) asserts that writing is 'the product of a creative, dynamic learning process that spirals naturally upward and outward toward limitless possibilities' (p. 8). All of these notions – inner speech, internal dialogue, and the social and dynamic aspects of learning – tie back to Vygotsky's (1978) underscoring of language as a social tool.

Writing's role in self-regulation: Bridging to individual thought

Writing research is increasingly drawing on Vygotsky's (1978) sociocultural theories (Burkhalter, 1995; DiCamilla and Anton, 1997; Everson, 1991; Freedman, 1995), suggesting that sociocultural and mental activity are connected in a 'dependent, symbolically mediated, relationship' (Lantolf and Pavlenko, 1995). Vygotsky's (1978) notion of higher mental functions occurring socially – on an external stage – before becoming a truly internal function, is a recursive process, as cognitive development occurs throughout one's entire life. As DiCamilla and Anton (1997) write:

> One of the main areas of inquiry in sociocultural theory focuses on how language serves to mediate human activity both on the inter-psychological plane in the form of social speech and/or writing and on the intrapsychological plane in the form of inner speech, which is externalized in cognitively difficult tasks as either private speech or private writing (p. 613).

In the latter domain, the authors report, the substantive content of speech and writing and an individual's ability to direct, plan, and guide himself through a variety of tasks, has been investigated (DiCamilla and Lantolf, 1994; John-Steiner, 1985; Pellegrini, 1981; Wertsch, 1979). DiCamilla and Anton (1997) remind us that it is through collaboration with others (collaborative discourse) within the interpsychological plane, that individuals become able to self-assist as part of an internally self-generated cognitive plan. They state, 'Language is used throughout one's life to regulate others and to regulate ourselves. It serves as a psychological tool in organizing functions that are critical to mental activity (e.g., voluntary at-

tention, perception, planning, memory, conceptual thought, evaluating)' (p. 613). In this paper, one feature of language – writing – is highlighted as a bridge to self-assisted individual thought, the point where the learner becomes self-regulated in the planning, performance, and evaluation of some functions.

Outside in – Inside out: The zone of proximal practice

The preceding theoretical discussion makes clear that writing involves a process that promotes learning. Writing requires students to use higher cognitive functions such as analyzing and synthesizing information. It helps develop students' thinking, requiring them to revisit their thoughts to organize information in ways that were not readily visible to them before entering the intrapsychological level. It is at the intrapsychological level in which independent thinking and problem solving are further developed. To this end, a new zone, the Zone of Proximal Practice (ZPP) is created.

For example, in a study conducted by Doolittle (1991), a sixth grade middle school computer class participated in an activity aimed at integrating writing with computer science. Taking approximately five class periods, the activity consisted of three phases: (1) the 6th-grade students wrote a story, using paper and pencil, in any literary format of their choice; (2) 11th- and 12th-grade volunteer editors from a word processing class helped the 6th-grade authors edit their stories; and (3) the editors and authors worked collaboratively to enter the text of the story into a desktop publishing program. During this editing phase, the authors worked closely with the editors. The editors were able to demonstrate many skills the authors had yet to acquire. Thus, the editors provided instruction that was in the authors' zone of proximal development. The editors, while not experts, were superior to the authors in terms of their knowledge of editing, and were able to stimulate the authors within their zone of proximal development. The core of the ZPD was the collaborative discourse between author and editor–a social system that is actively constructed, supported, and scaffolded by the authors' interaction with the editors. The stories became more intricate and less in need of editing as students repeatedly participated in the collaborative activity. The Zone of Proximal Practice (ZPP) was the intradependent medium in which this happened.

In Figure 1, Vygotsky's ZPD construct is amplified to reveal the zone of proximal practice – the zone characterized by self-regulation. The ZPD is the context in which the individual learns in collaboration with others, is represented as 'Outside-In,' in which social interaction and 'other-assistance' is embedded. What the individual internalizes and manifests with the help of written language becomes part of his understanding ('In-

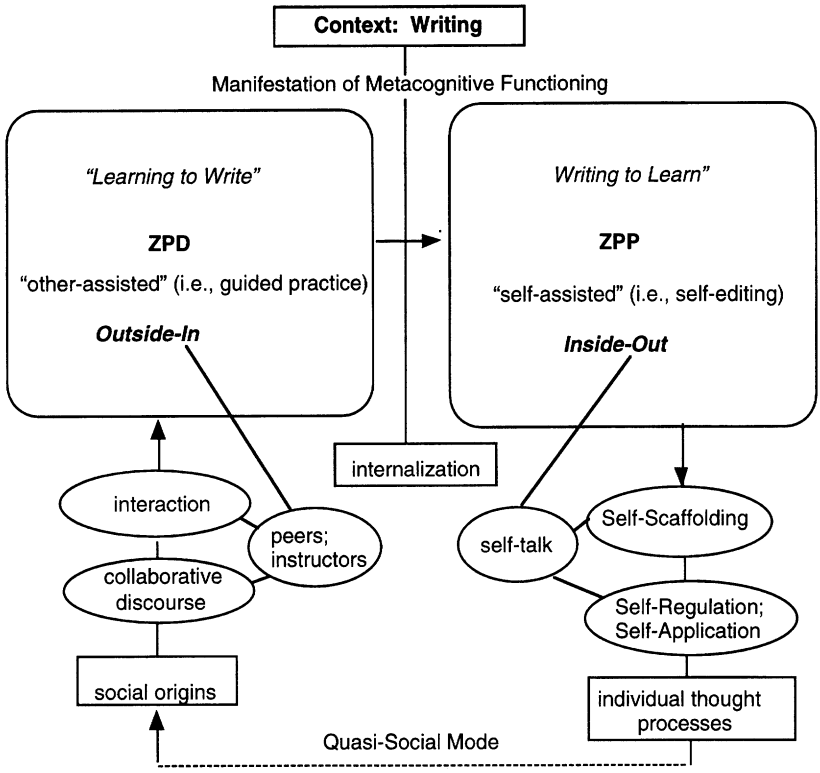


Figure 1. The ZPP: Amplification of Vygotsky's ZPD Construct.

side-Out') within the zone of proximal practice. The ZPD and ZPP are conceptualized on a horizontal plane, where the two interweave in a quasi-social mode (Vygotsky, 1978; Wertsch, 1985). In Doolittle's (1991) activity, the authors' increased ability to apply editing skills to their work is an example of self-regulation – internalizing and applying the principles of editing to subsequent writing. The writing process, therefore, was used as a tool for mediating cognitive development, as evidenced by the authors' ability to self-assist within the ZPP.

Vygotsky (1981) proposed that in order to understand individual development it is imperative to understand the social contexts in which the individual resides. Individual development involving mediated tools undergoes qualitative changes when it 'transitions from a social to the individual function' (p. 159). The ZPP represents an attempt to capture basic understanding of mediated tools in this transition. The ZPP provides an important zone within which students can transform learning developed from social interactions and use that learning as a catalyst for independent thinking and problem solving. Students move from a practice in which they

TABLE I

The relation between the Zone of Proximal Development (ZPD) and the Zone of Proximal Practice (ZPP)

	From discourse to writing	From inter- to intrapsychological	From new insights to practical application
Interview	Oral thought processes	Collaborative problem solving	ZPD
Writing samples	Written thought processes	Individual problem solving	ZPP

have been receiving assistance (i.e., ZPD), to one in which they practice unassisted (ZPP).

The ZPP and problem solving. Oral language is the tool used to shape the discourse in collaborative problem solving; however, at an independent level of learning and development, writing is the tool students can use to shape their thinking. Thus it is 'clear that the larger setting in which activities occur plays a crucial role in shaping the structure of the activities, individuals' goals, and the constraints on achieving those goals' (Cole, 1996, p. 341). Put another way, writing is as much a tool used to express mathematical thought, as it is a product of collaborative problem solving. 'Written speech is monologues; it is a conversation with a blank sheet of paper' (Vygotsky, 1986, p. 181). It is the inner conversation with 'self' that explicitly affords students opportunities to write, practice, and make their thinking visual and concrete.

Central to understanding the interconnection of the ZPP and the ZPD is to think of practice as the context that gives form to nondescript mathematical thought. Cole (1996) writes that in 'practice we at least have an opportunity to put different interpretations into dialogue with each other, and thereby, to learn more about each voice in the dialogue' (p. 343). In the ZPP, the dialogue with self provides the collective voice that has been grounded in collective practice with others. In this zone, writing provides another opportunity to acquire an understanding of relationships and ideas introduced or explored within the boundaries of collaborative situations. It is a very important source because the writing used by students when describing how they solved a problem reflects their personal knowledge of the collaborative activity. The collaborative discourse influenced the writ-

ing discourse. This construct is illustrated in Table 1, showing the relation between the ZPD and the ZPP.

Building on the research of Vygotsky regarding the role of social relations in learning and development, the study described in this body of work attempts to extend our knowledge of students' mathematical thought construction. This investigation focuses on the ways in which seven students engaged in a socially-constructed problem solving activity, using that activity as an infrastructure in which to build their understanding of mathematics through proximal practice. The new zone that emerges is the result of the students' transformation from the interpsychological to the intrapsychological plane of functioning. A basic assumption that underpins the ZPP construct is that the writing students do, as shaped by the collective practice in which they engage, determines how they independently think about mathematical ideas or concepts. From this perspective, the ZPD furnishes a social context for students' thinking during collaborative problem solving, in which their understanding of that social context is further developed through writing. Vygotsky (1986) suggested that writing is a tool for expressing inner thoughts and understanding; thus 'written speech ... must explain the situation fully in order to be intelligible' (p. 182).

The theories and ideas discussed in this section reflect and support the assumption that students must go beyond the information given to transform mathematical concepts and ideas into a more useful personal structure of knowledge (Bruner, 1973). It has also been suggested that oral and written statements are important for the development of students' cognitive thinking. This understanding has helped ground the present study, shaping the position that writing in the 'zone of proximal practice' is an inclusive process that provides practice that develops thought inside while taking advantage of outside social contexts.

3. Context: The Larger Study

The ideas presented in this paper are part of a much broader and more extensive work that explored the evolution of thought processes in a seventh-grade classroom (Albert, 1995). The fourteen-week study provided a picture of how the development of cognitive processes is enhanced in the classroom when instruction and learning rely less on traditional approaches (e.g., 'chalk and talk'). The intent was to move towards approaches that included collaborative group activities in which students learned mathematics by gradually turning interpersonal speech (group talk) into intrapersonal speech (self-talk) – which in turn helped them to solve problems. This study described how problem solving became the center focus of

the classroom curriculum. The approach employed for the study was an adaptation of Polya's (1945) problem solving model. Following Polya's framework, the adapted and expanded model combined three strands that directly connected problem solving practices to writing activities, describing the action of the teacher, the action of the students, and the assessment of students' action. Each strand consisted of three inter-related parts: (a) understanding the problem (e.g., actions the teacher and students take before solving the problem); (b) choosing and implementing a solution strategy (e.g., what the teacher and students do to actually solve the problem) and (c) getting a solution and evaluating it (e.g., actions the teacher and students take after solving the problem). The pedagogical practices employed by the teacher involved a class discussion of the problem. Next, while the teacher moved among the groups, assisting students, asking questions, and making recommendations, the students worked collaboratively in groups where they discussed and took notes regarding strategies and procedures necessary for solving the problem. The students then worked independent of their group to write statements, explaining the strategies and procedures they employed to solve the problem. During the next class session, students came together to discuss their solutions to the problem.

The study used a mixed methodology design that included three comparison groups including both quantitative (quasi-experimental) and qualitative (case study) approaches. The teacher, who had 14 years of teaching experience, taught all three classes. The practices of solving problems were studied intensively using a variety of data sources: interviews (teacher and students), classroom observations three times per week for 14 weeks, pre- and post-problem-solving tests, and pre-and post-attitude questionnaires (attitudes about writing and about mathematics in general). The findings of this study indicated that students in the experimental group showed significant improvement in their understanding of problems and in their problem solving performance compared to the students in the control groups. In addition, the students in the experimental group tended to have improved or positive attitudes about writing to learn mathematics over the students in the control groups.

The study provided some understanding about classroom practices that were complex for students, as well as understandings concerning the teacher's conception of teaching and learning of mathematics. The teacher changed the way she taught mathematics, shifting from a traditional textbook and skill-driven approach to a personalized problem solving approach. The teacher's beliefs, preferences, and views about how students learn mathematics played a consequential role in shaping classroom discourse and student learning. The teacher found a path that led to critically ques-

tioning her beliefs and their affect on student learning. The teacher adopted the position that her role as the teacher called for making problem solving the focus of classroom instruction.

4. Methods and Procedures for the Present Study

An interpretive case study approach was used to provide documentation about the seven students that participated in this study. Case study research helps readers in the construction of knowledge and provides some insights about a single case surrounded by issues or themes studied (Stake, 1994). This study presents the seven students as a single case, rather than individual cases. Limiting the number allows exploration of the case to a reasonable depth within the scope of time and other resources that are available for this study. In particular, this interpretive case study is an exploration of a bounded system. 'The bounded system is bounded by time and place, and it is the case being studied – a program, an event, an activity, or individuals' (Creswell, 1998, p. 61). This study attempted to explore how a socially constructed context was related to a specific individual practice (i.e., writing) in which students engaged. Vygotsky (1978) argued that research should result in dynamic analysis in which 'the complex reaction must be studied as a living process, not as an object' (p. 69). Thus, it is important to study processes leading to outcomes. In this case, the focus was on the seven students as a single case to provide information about how a social practice (different from the classroom social context) informed independent thinking and problem solving.

Setting and students

The seven participants were selected from the larger study of about 60 students from three seventh-grade mathematics classes located in the Midwestern part of the United States. The middle school that the students attended had an enrollment of about one thousand students from various socio-economic and cultural backgrounds in grades six, seven, and eight. About 67.5% of the students were white, 25.2% were African-Americans, 1.9% were Hispanic, 6.3% were Asian or Pacific Islanders, and less than 0.3% were Native Americans. Thirty-four percent of the students attending school were identified as coming from low socioeconomic backgrounds and 3.2% were classified by the school as Limited-English Proficiency students. The seven participants for this study consisted of four females and three males all from middle-class backgrounds. Two of the females were students of color (one African-American and one Asian-American). The

other two females and the three boys were white. The seven students were chosen because they were present for the entire fourteen-week period of the larger study, had actively participated in all problem solving classroom activities, and were all members of the same experimental class. In addition, special informed consent was given by the parents of these students to participate in the interviews with the investigator. Pseudonyms are used to protect the confidentiality of students. LA represents the name used by the investigator/writer.

Data collection procedures and analysis

There are two primary data sources used in this study: interviews and samples of students' writing. The interviewing of students took place during the final week of the larger study. The interview process was semi-structured, using a prepared list of questions (See Appendix A). More than twenty-five writing samples per student were collected over the course of the study. For the purposes of this study, writing is used in a broad sense to include all written work, verbal and figural, i.e., diagrams, graphic representation (Ernest, 1998). The intent was to use these data sources to examine students' thinking processes rather than to determine whether students provided right answers.

Interview Protocol. The interview protocol consisted of 26 questions and focused on eight problem-solving themes: comprehension, approaches or strategies, relationships, flexibility, communication, examining solutions and results, mathematical learning, and self-assessment. The interview questions and the process were adapted from *Mathematics Assessment* (Stenmark, 1991, pp. 31–32). Calculators and manipulatives were available for students use. All interviews were audiotaped and transcribed verbatim for pattern matching analysis. The students, in two separated groups, were interviewed together for 50 minutes. Before beginning the interview, the students worked in their groups to review the problem and discuss strategies and procedures regarding how to solve the problem. The City Bus Problem (See Figures 1 and 2) was selected as the interview problem to serve two purposes. First, the portrayal of mathematics as the discovery of patterns is depicted clearly by the problem, which made it similar to problems the students previously experienced during the larger study. Second, this problem could be solved through the use of a graphic representation, which served as a bridging tool for facilitating pattern exploration.

During the interview, the investigator asked leading questions or provided hints to students. The intent was to assist and scaffold the problem solving activity to help students make progress as they co-constructed strategies or procedures to solve the problem (Burner, 1975; Vygotsky, 1978). The

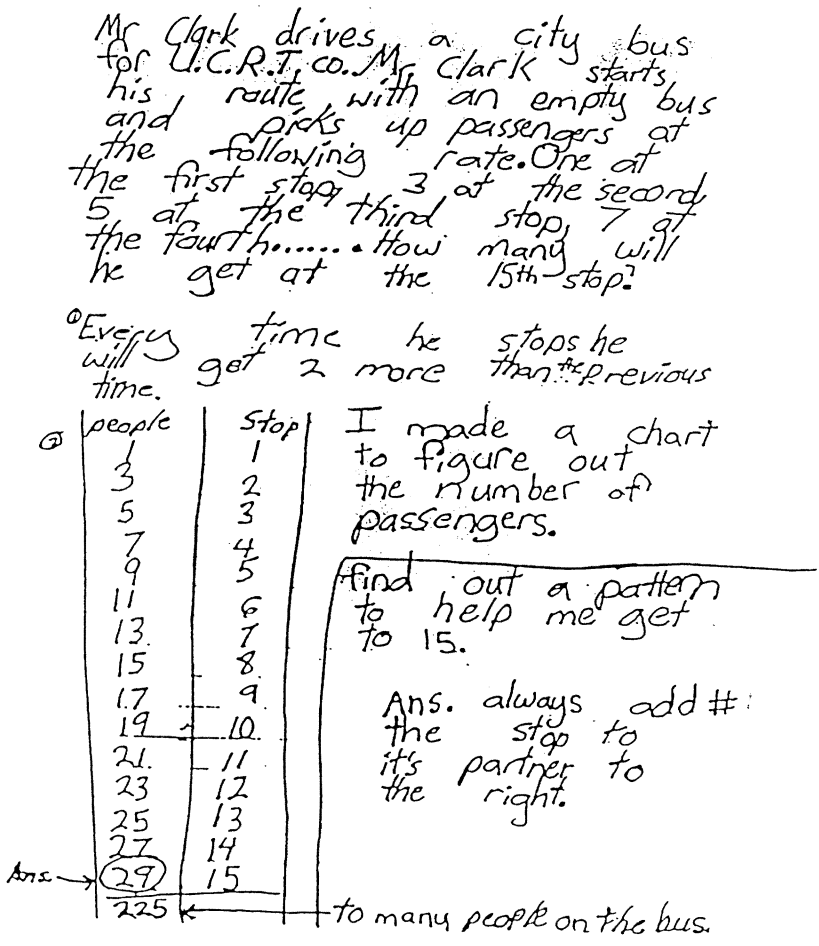


Figure 2. The City Bus Problem.

role of the investigator was not to simply gather information about students' problem solving understandings; rather the role of the investigator was to facilitate the learning process through questioning and offering suggestions that assisted students in making sense of the mathematics.

Writing Samples. Selected samples of students' writing were collected for analysis. These samples consisted of descriptions of problems students solved during the interview and samples from problems worked during the instructional phase of the larger study (See Figures 1, 2, 3, and 4). The writing samples for this study were selected because the writing was about problems that introduced new material not representative of common elementary mathematics (e.g., computation skills that focus on the four basic operations). For the larger study, students were instructed to rephrase the

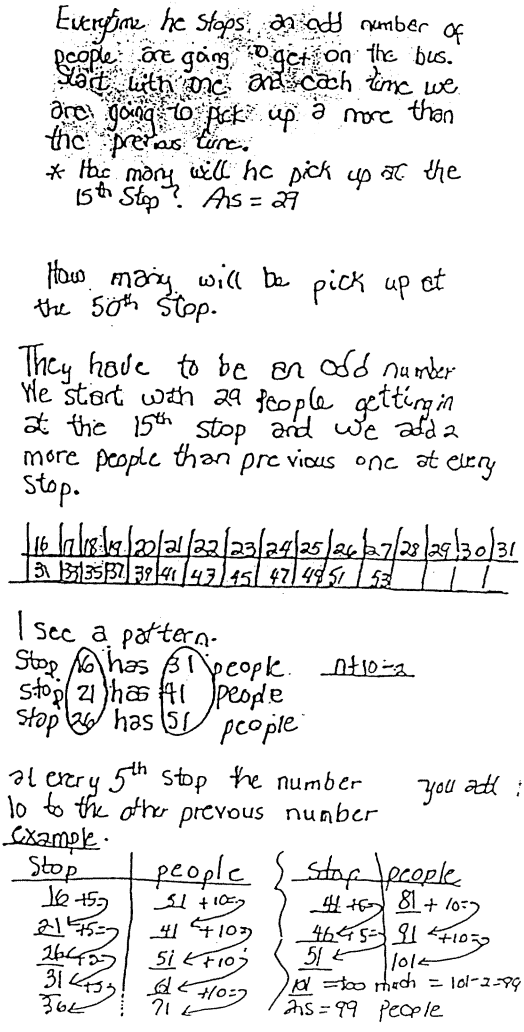


Figure 3. The City Bus Problem Revisited.

problem in their own words and to write descriptions of their reasoning, procedures, and strategies or approaches used to solve each problem. During the interview, students wrote notes about ideas and strategies discussed, but they did not write constructive statements until the conclusion of the interview.

Data Analysis. Analysis encompassed both transcripts of interviews and the writing samples. The unit of analysis for the interview protocol was the presence of the problem-solving themes: approaches and strategies, relationship, communication, examination of solution, and mathematical learning. Comments on each theme were coded into major related cat-

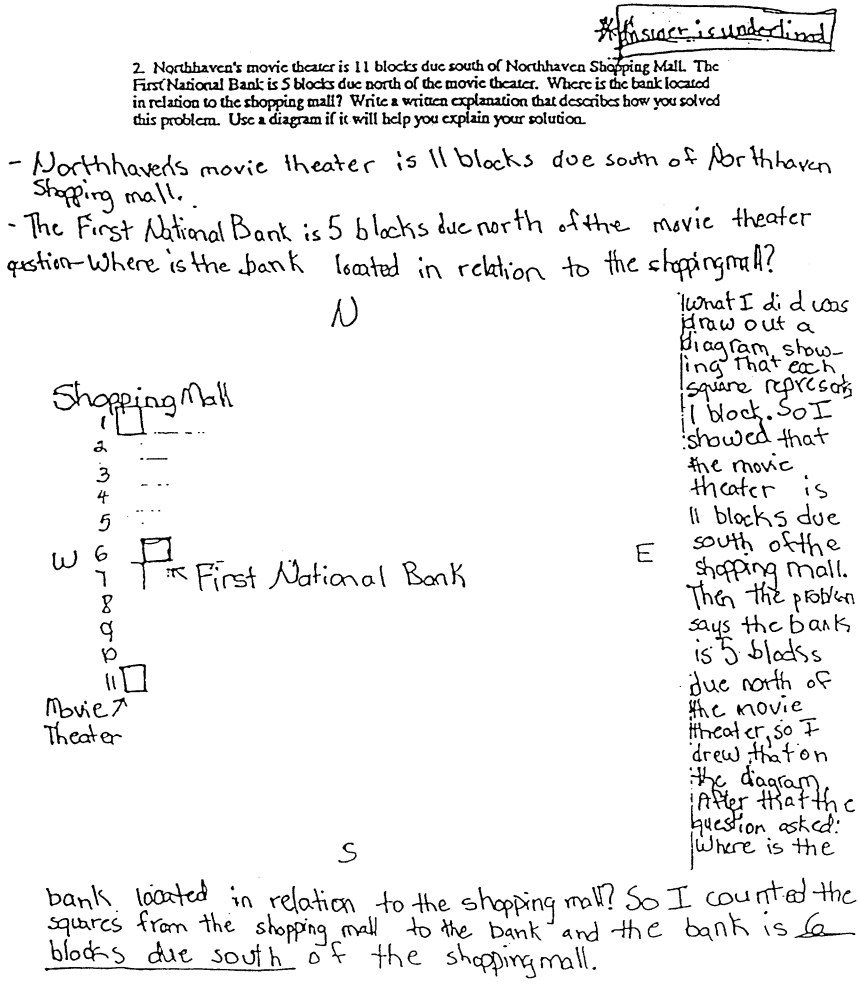
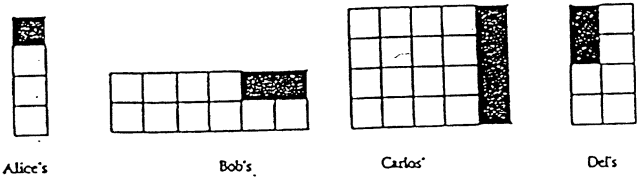


Figure 4. The Shopping Mall Problem.

egories. The 'textual analysis' (Silverman, 1993) of the students' writing provided further insights into their conceptual understanding of the problems presented to them. Other factors considered in analyzing students' oral and written responses included illustration of understanding of underlying concepts presented in the problem, selection of an appropriate strategy to solve the problem, completeness and accuracy of a reasonable solution to the problem, and oral and written explanations or statements articulating how the problem was solved.

Limitation of Findings. It is clearly impossible in a single paper to incorporate all the issues and perspectives surrounding this study. Consequently, the presentation of findings in the next section is highly selective; the

1. Directions to the Student: This is an open-ended question. Your answer will be judged on how well you show your understanding of mathematics and on how you can explain it to others. Please write your response in the space below the question.



These are the floor plans of four different restaurants. The restaurant owners are meeting to discuss how they are providing areas for smoking and non-smoking customers. The shaded areas represent smoking areas; while the white areas represent non-smoking areas. Bob says that he has the greatest fractional part of his restaurant for smokers. Alice, Carlos, and Del disagree with Bob. Please settle the argument and help each owner to understand your answer. Use a diagram if it will help you to explain.

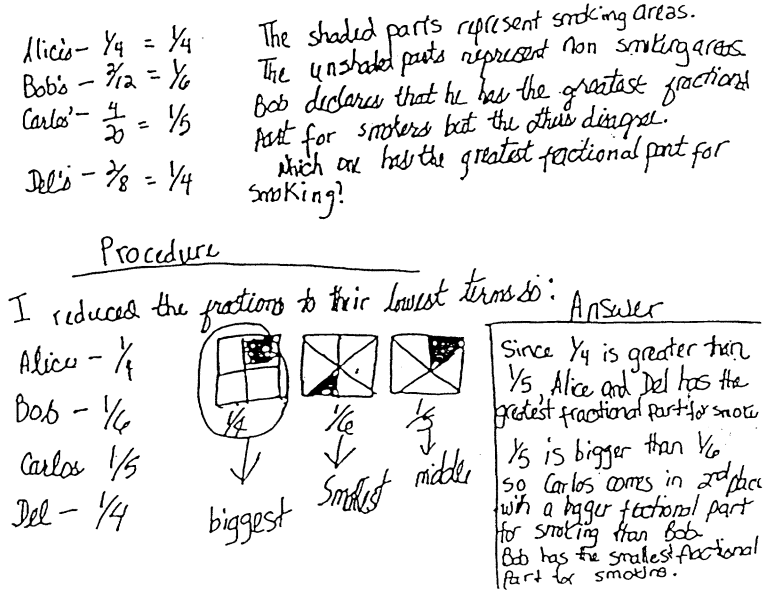


Figure 5. The Smoking Area Problem.

author acknowledges that many issues and meanings of collaborative discourse, problem solving, and writing are not covered here. It is recognized that the interpretations upon which the findings are based targeted a small sampling of students. This study's findings, however, do suggest the importance of creating new learning environments that promote and connect learning and development.

5. INTERPRETIVE FINDINGS

Connecting social practices to independent practices provided a deeper and broader understanding about students' mathematical knowledge construction. This information offers a description of students' beliefs about writing in a mathematics classroom. The findings are reported in case scenarios and episodes that occurred during the interview protocol. This section also includes findings based upon analysis of students' written explanations and presents several insights about students' problem-solving practices. The following scenarios are organized around three emergent themes from the data analysis: (1) This Is Easy; (2) Anybody Can Add; and (3) The Math is Right. The interview discussion focused on the following problem:

Mr Clark drives a city bus for Union City Regional Transportation Company. Mr Clark starts his route with an empty bus and picks up passengers at the following rate: one passenger gets on at the first stop, 3 get on at the second stop, 5 get on at the third stop, 7 get on at the fourth stop, and so on. How many people get on the bus at the 15th stop?

This is easy: Keeping track of my thinking

Before beginning the interview, students worked together to explore the problem stated above and to discuss the important information regarding strategies and procedures needed to solve the problem. Then, as a starting point for engaging students in conversation, the interview began with a request to explain how they organized the mathematical information in the problem:

Ann: I made a chart. I have the number of stops and underneath I placed the number of people getting on at that particular stop.
LA: Why did you organize it using a chart? Ann: I don't know.
LA: If you had not organized the information in a chart, what do you think might have happened when you tried to solve the problem?
Ann: I would have gotten it wrong.
LA: Why would you have gotten it wrong?
Ann: Well, organizing it and writing it on paper sort of helped me keep track of what I was thinking.
LA: Angie, I see that you used a chart also. Why did you decide to use it?
Angie: Because it was kind of easy.
LA: Why was it easy?
Angie: Well, I just wanted to do something different.
Dana: I know why. Because it made it easier for me to get the answer.
Chen: It's easier because it (the writing) helps you understand the question.

The students articulated their reasoning for using a chart to organize the information posed in the problem. (See Figure 1, the City Bus Problem, for

an example of the types of chart students constructed.) Students explained that a major motivation for using the chart was to get a correct answer; their responses indicated that the use of the chart made it less difficult for them to solve the problem. The chart served as a communicative tool for assisting them in analyzing their understanding of the problem. For example, Ann explained how she constructed the chart, but when asked why she used a chart to organize the information, she stated that she did not know. Further questioning by the investigator provided the mediation that Ann needed to build her understanding. She was then able to present an argument for using the chart. The degree of guidance or scaffolding needed by the students varied with each student. For instance, Dana and Chen needed little, if any, prompting to answer the question. It is suggested that the guidance received by Ann and Angie was consciously acquired by the other two students.

Students also connected the writing to the strategy used to solve the problem. This connection clarified and extended their thoughts as they reflected on the experience of using a chart to organize the information presented in the problem. Next, the students were asked to identify a pattern. The scenario continued with the students describing how the chart helped them identify the existing pattern.

LA: Look at your chart. Is there a pattern?
Students: Yes.
LA: What is the pattern and what can you tell me about it?
Ann: You have two numbers $1 + 2 = 3$. You get odd numbers like $1 + 2 = 3$, $3 + 2 = 5$ and $7 + 2 = 9$. There's always 2 more people getting on the bus.
Dana: Every time the bus stops, two more than the previous got on the bus. Is it $n + 2$?
LA: What do you think, Dana? Think about what Ann said.
Dana: n would be the previous stop. So, it must be right.
LA: What does the two represent?
Dana: The number of extra people getting on each time.
LA: Look at the pattern. Can we come up with an equation to solve the problem?
Angie: We can use n times something equals. But, I can't explain it.
LA: What would you multiply n times?
Angie: n equals 2. At the first, stop if one person got on the bus that would equal 4. But we said that it would be an odd number.
LA: What do we need to do?
Ann: Subtract 4 minus 1. That's 3.
Angie: Three is an odd number. It would be 2 times n minus 1 equals.

Students identified the pattern and the constant difference. Investigating the problem in the context of finding a pattern provided for broader explorations of the formation in which the pattern existed. Thus, the investigator's response to Dana was not to indicate to her that she was wrong, but to probe her further regarding her solution to the problem. In essence,

Dana deduced a recursive description of the sequence when the investigator asked for a general or explicit one. She constructed an appropriate recursive description of the pattern of the sequence when she gave the n^{th} term as a computable number of the previous terms. The intent was to scaffold the discussion in such a way that students would come up with a general formula for the 15^{th} and later the 50^{th} term. Dana's description informs us that, no matter where we are in the sequence, we can get the next term by adding two to the term we have.

With Ann's help, Angie discovered for herself the equation and with some probing, she articulated her understanding of that equation. Thus, the general description of the pattern was co-constructed by Ann and Angie. Further analysis indicated that Ann and Angie's responses were based upon information supplied by Dana. Dana's explanation helped them to make connections between the original problem and the equation. Another student, Chen, who sat quietly through most of the discussion, stated that we could use the equation because it was a better approach. When asked why using an equation was a better approach, Chen suggested, 'the equation made the problem easier to solve'. Therefore, the questioning of students launched and assisted them in developing ways to communicate their mathematical knowledge and understanding by using practices rendered by other group members.

Anybody can add: Students' attitude about writing

The inclusion of writing calls for students to examine solutions or results to evaluate the reasonableness of an answer with the goal of helping them focus more on process and less on getting the correct answer. To exemplify how students communicated their understanding of procedures or strategies used to solve the 'City Bus Problem', I asked: How did writing help you explain your thinking? Overall, the students expressed that writing helped them to better understand the problem:

Chen: Well, it helps me explain what I am thinking.

Mike: The writing helped me a little bit. It helped me understand a little bit more. [It helped me] get a clearer picture of what's going on in the problem.

Dan: Using the chart and writing sometimes made it a lot easier to figure out the answer. Then you just write it out. It's basically just doing what you do normally. You [are] just using writing.

Dan's comments indicated that perhaps he drew upon his experience. According to him, this mode of discourse (writing) is not different from what he normally did when he solved problems. In other words, using writing was just another way of expressing ideas. It was not something new, thus, it should not elicit anything special. Dana gave this well-constructed

answer: 'I think writing helps some people, but some people can just do it in their head and figure it out just like that . . . . I can't do the problem in my head. Writing the problem out helped me. It helped me know the questions'. On the other hand, some of the students communicated their uncertainty about writing when attempting to answer the question. It was not an easy process for students to explain how writing deepened their understanding. Sometimes, students made personal connections to explain what they were thinking. Angie cited this example:

When you write something down and you forget what you wrote you can go back and read it. It really would help in the future. Like my mom, she [did] things when she was younger and she's older now. She can't remember some of it because when I bring home my stuff, she remembers it. She sees it. I think writing it down helps her remember.

Building on the above statements about how writing helped them learn mathematics, I then asked the students how they would explain the process used to solve the problem to a fifth-grade student. The objective was to develop a sense of the students' understanding of the problem. Their explanations disclosed that they had fixed ideas about how to share information with younger students and how to help them construct their understanding of the problem. For instance, the students proposed that they would share ideas, give clues, and suggest a strategy (for example, making a chart or table) to the student, but they would not give the student the answer. The ideas expressed here by students illustrated the importance of focusing on the process of problem solving rather than on the products generated. The students' explanations modeled how the students would assist the learning of a younger child. Students' reflections demonstrated that they understood the importance of facilitating each other's learning.

Dana and Angie's comments revealed that they would read the problem to a younger student. It was not difficult to understand why they wanted to do this, because reading the problem before solving it was directly connected to their problem-solving experiences. Their teacher often read the problem to them during the early phase of the project. However, it is possible that students felt the problem would be too difficult for the child to read and understand because a younger child would be several grades behind them. The students learned earlier in the project that understanding the information posed in problems made ideas clearer and helped them find appropriate strategies that led to reasonable solutions. Students expressed the following:

Dana: I would read them a problem and give them some ideas on how to solve the problem. I don't really tell them the answer, just give them some ideas on how to solve the problem.

Angie: Read them the problem. Then give them some clues and help them solve it. I'd start them out with a chart and go halfway down, [then] let them finish [it].

Dan: The chart you used shows them basically, how you did it . . . . I mean you can make it quite easy.

Mike: I would just show them how I got the answer. It's pretty simple. I would tell them to say like 1 + 2 is 3. [Be]cause anybody can add; so we would know how to keep going until the 15th stop.

Dave: I'd probably make a chart of stops like I did here. (He is referring to his chart.) Like Dan said, anybody can add. Give the first three and see if they can figure out how it goes on up and I could . . . .

Dave was not satisfied with his explanation. He added, 'I could try to explain it better'. When asked how to explain it better, Dave announced after some hesitation that he did not know how to do it. However, Dave's answer did suggest that he was questioning whether his explanation of how to use the chart would be instructive for a younger student. Dave and Dan also assumed that, 'Anybody can add'. Conceivably, students' thinking may be evolving toward the realization that conceptual understanding, ideas, and strategies for problem solving are essential to mathematical learning. Students also grasped the idea that basic computational skills are not necessary for conceptual understanding. These students preferred to focus on explaining ideas and strategies rather than focusing on computational skills. That is, it would be appropriate to give students ideas and clues. It would not be appropriate to give the answer nor to show younger students how to add because, 'Anybody can add'.

Hey! The math is right but the answer is wrong

The following conversation began with an extension of the original problem. I asked students how we could determine the total number of people on the bus. The original problem only asked for the number of people getting on at the fifteenth stop. Students responded in this manner:

Mike: We'll just have to add the number of people who got on the first stop to the second stop and so on and that will give us the answer.

Dave: We need to figure out the population.

LA: What was the total number of people on the bus at the fifteenth stop?

Students: 225.

These students illustrated an appropriate understanding of the mathematical element in the problem, but their solution was flawed. Students' flawed solution arose from the problem's structure and context and not with the students. The plan was for students to construct mathematical ideas as they explored real-world problem situations that proposed a dilemma. Their conversation led them to the insight that one bus could not hold 225 people. Dan demonstrated this when asked if he thought 225 people was the only possible answer. He stated that it was the only right answer because:

[I]t gives you all the information [needed] to answer the specific question, how many people got on at the fifteenth stop. If it had said how many extra people got on . . . .

After some thought, Dan proposed this solution to the problem:

Dan: Unless it's like a French bus that has a top piece on it.
LA: Even with a French bus, do you think it would hold over 200 people?
Dan: No, it'll be a few buses, not just one.
Dave: Could be easier on a train. It's not realistic. The bus is small. The train is bigger.
Mike: The bus driver got to be mad or something. 'Get off my bus!' Could be easier on a train.
LA: Why is it easier on a train?
Dave: Because, like there's so many people on the bus and it's small.
LA: Have you ridden a city bus before?
Dave: Yes.
LA: How many people do you think the city bus can hold?
Dave: Sixty sitting down and about seventy standing up. You almost have to strap them to the roof.

Students were able to propose a solution and offer reasons for their proposal. Dave clarified his explanation by giving an estimate of the number of people he felt the bus could hold. During this phase of the interview, Mike suggested, with humor, that the bus driver was angry and ordered some of the people off the bus. He also suggested that some people could possibly be strapped to the roof of the bus. I did not challenge or question him about these suggestions. I simply let them stand as possibilities because I wanted to see if he would elaborate on these possible solutions. This dynamic engagement on the part of the students generated an effective solution for a flawed problem. Through appropriate scaffolding, the students successfully judged the quality of the solution and provided a more effective alternative for the problem.

To explore the students' understanding of the connection between this problem and real-world situations, I asked Dan, Dave, and Mike how the information posed in the problem could be used in everyday life. They stated:

Mike: If you are in a town trying to figure out how much money should have [been] taken instead of how much money they have gotten.
Dave: The bus driver might have taken some money and they want to know if they have a problem with it.

Again, students' comments indicated that they could connect the original problem to real-world situations. The alternate solutions and the realistic connections provided a momentary view of the students' metacognitive knowledge as it aided their understanding of this problem and the application of their mathematical knowledge to the real world.

Writing: Communicating understanding

An integrated composite of writing samples (including selected samples of problems completed during the classroom instructional component of the larger study as well as from the problem discussed during the interviews) depicts how these students used writing to communicate mathematical understanding. The following results include actual examples of students' work to illustrate the students' use of writing to express their thought processes. Students' written explanations presented four distinctive insights about their problem-solving processes. The insights included the following: understanding of problems, strategies or procedures used to solve problems, observed patterns and relationships, and evaluation of the reasonableness of solutions to problems.

Figure 1 is an illustration of the 'City Bus Problem' worked by students and Figure 2 is an extension of this problem. Each example shows thinking that involves reasoning, exploring and processing information. The students appropriately paraphrased the problem and identified the important elements of the problem. For example, each student wrote that there had to be an 'odd number' and that there must have been 'two more people than previous stop'. This problem was explored by students during the interview in which the primary function was to help students develop mathematical ideas and to study the mathematical structure, i.e, patterns. Much of the writing about this problem occurred at the conclusion of the interview, independent from other group members. However, analysis of this problem showed that working collaboratively helped students develop their written responses to the problem. For this problem, each student went from collective practices (interpsychological) to independent practices (intrapsychological) in which they were able to write about and make sense of the mathematics explored in the group. Another way of thinking about this construct is that both practices are communicative tools that mediated the students' learning of mathematics. Using Wertsch's (1991, 1997) 'tool kit analogy ... with the understanding that different groups may employ similar tools in different ways' (p. 95), writing was merely one tool that students had in their tool kit; it was one of many different ways of learning and understanding mathematics that the students employed. Individual students approached the problem by connecting with the collective activity (the interview in this case) to shape their writing.

Students were able to make conjectures about the strategies or procedures used to solve problems as evident in Figures 1, 2, 3, and 4. For Figure 4, the 'Smoking Area Problem', the student explained and tested his conjecture about how to solve the problem. He simplified the fractions, compared them, and emerged with an understanding of the problem

concept as well as the mathematics presented therein. It is also evident that the pictorial representation of the fractions helped the student reach a solution to the problem. The visual models of the fractions one-fifth and one-sixth, although incorrect, serve as placeholders for those fractions. The student disregarded equal size or shape and analyzed the fractions as pieces or parts (i.e., one piece of five and one piece of six pieces). Thus, the student's understanding of the problem was enhanced not necessarily by the quality of the models, but by the inventiveness of a model that served his cognitive needs.

In Figure 1, the City Bus Problem, this student incorporated an appropriate strategy and included a summary statement about the chart. The chart as a visual representation of the problem not only helped students organize the information into a meaningful structure, but also helped the student see an important aspect about the problem: this student concluded that too many people were on the bus. This conclusion was explored during the interviews. Students suggested that the mathematics for the problem was appropriate, but the answer was wrong because it was not reasonable. This instance of evaluating the reasonability of an answer serve to illustrate the development of higher cognitive functions.

Additionally, the chart helped the students to perceive the pattern that was obscured in the problem structure because it was a dynamic representation of the problem, assisting them in establishing a chain of thoughts in solving the problem. For example, in Figure 2, the student who wrote the expression $n + 10 - 2$ to represent the pattern for every 5th stop discovered and included an explanation about how to use the pattern. The student gave several examples of how to use the pattern to solve for the problem. However, the student in Figure 1 noted that a pattern would help him obtain a solution but did not write a numerical expression for the pattern. Students appeared to be aware of the relationship between the pattern and the information needed to solve the problem. These mathematical ideas were discussed during the interview. The group work served as an external mediated tool for developing self-understanding of the problem.

Finally, the writing revealed the process that students employed to evaluate the reasonableness of a solution. In Figure 3, the Shopping Mall Problem, the student referred to the questions asked in the problem. The student wrote, 'The question asked where is the bank located in relation to the shopping mall? So I counted the squares from the shopping mall to the bank and the bank is 6 blocks due south of the shopping mall'. The written statement for the Smoking Area Problem demonstrates how that student reasoned through the information to reach a solution. This student not only explained who had the largest fractional smoking area,

the student's application of this problem also illustrates how and why 'Bob has the smallest fractional part for smoking'. In both cases, the students followed a systematic procedure for solving the problems and the writing served as the mediation tool that assisted the students in transforming the problem. Thus, the students became metacognitively aware of their own thinking and learning.

Overall, the examples provide a glimpse of the students' insight into and understanding of the problems presented to them. They were able to identify and explain appropriate strategies needed to solve problems and to make some conjectures about patterns and solutions obtained. Not all seven students, however, were able to express themselves in writing at the level illustrated in the writing samples. However, further analysis suggests that all seven students frequently restated the problem into their own words and used diagrams, charts, and other graphic models to obtain reasonable solutions.

6. Discussion

Reconstruction of one's knowledge comes about in many ways. In this study, students accomplished this reconstruction in two major modes: initially, in communication with their teacher and peers, and subsequently, in writing. This reconstruction of one's knowledge can be construed as the acquisition of individual thought processes. Building on Vygotsky's notion of the role of social interaction and the zone of proximal development in learning and development, a new zone was proposed. This zone, conceived as the zone of proximal practice (ZPP), provides the reader with a context for understanding students' evolving thinking, as illustrated by self-regulation and self-assistance. This discussion reexamines some of the major findings, which consider particular issues, and insights that emerged in the interpretive case study and in the analysis of students' writings. It includes an interpretation of findings with the theoretical assumptions discussed in this paper through exploration of the relation between oral conversation and writing as communicative tools for articulating mathematical thoughts and ideas.

Articulating mathematical thought processes

Interviewing and observing the students provided evidence of how they articulated their understanding of mathematical concepts through both oral conversations and written explanations which represented their thought processes. Students engaged in collaborative discourse, which required them to focus on process and to conceptualize knowledge in a context that

made sense to them. They were encouraged to think through problems and identify necessary information needed to solve problems, followed by written explanations illustrating their thinking and learning. Students used a variety of strategies to solve problems and usually wrote statements to explain strategies or procedures used to solve those problems. This practice required active thinking on the part of the students, who used 'self-talk' as part of their analysis. Vygotsky (1986) argued that this type of higher level thinking is the result of active learning in social settings that leads students to a point where they can develop and self-regulate their progress. That is, students demonstrated increased proficiency while controlling their work independently, generalizing understandings across content area, and organizing information and resources to write explanations about strategies and procedures used to solve problems. Furthermore, students metacognitively directed their writing, as evidenced by the students' writing samples. In directing this process, the social activity should not be taken for granted because cognitive functions are shaped from participating in socially organized practices (Cole and Scribner, 1981; Scribner, 1997).

The importance of the social context of learning has been examined by other researchers as well (Lave and Wenger, 1991; Lemke, 1988; Russell, 1997). Lemke analyzed a classroom episode to illustrate the relationship between oral and written communication and argued that all modes of social interactions are mediated actions. Lave and Wenger, for example, wrote that 'Knowledge of the socially constituted world is socially mediated and open ended. Its meaning to given actors, its furnishing and relations of human with/in it, are produced, reproduced, and changed in the course activity' (pp. 50–51). The present study provides further insights regarding these cases. The writing in the City Bus Problem evidences that students reconstructed their understanding by using ideas and strategies discussed during the interview.

Bruner and his associates (1966) viewed the writing process as the basis for concept formation and as a tool for cognitive growth. Hence, writing, as a tool, supported the students' thinking and helped them construct knowledge for themselves about mathematical ideas. These students were encouraged to find their own paths to solutions, to connect them to their own ideas, and to develop a sense of confidence in solving problems on their own. The interview of students represented an attempt to capture the salient features of transforming learning from the social to the personal. It depicts students' shared meaning of mathematical ideas. This perspective suggests that the development of understanding of mathematical concepts is enhanced when students communicate their understanding through problem solving practices by which they are encouraged to discuss and write

about their ideas, interests, and experiences. The Vygotskian framework in which this study is grounded provided insights regarding this notion of inter- and intrapersonal learning. Learning cannot take place through activities performed by students in isolation of their social environment. Rather, as Vygotsky (1978) proposed, learning and development unfolds through discourse in relation to instructional practices mediated by peers, teachers, and other adults.

How does writing support students' understanding?

A large body of literature suggests that writing encourages students to think, reason, and utilize higher levels of cognitive thinking skills (Bruner et al., 1966; Elbow, 1981; Emig, 1986; Graves, 1983; Hoel, 1997; John-Steiner, 1985; Moffett, 1981; Smith, 1988; Vygotsky, 1978). The findings from this study imply that students' written explanations served as a starting point in the development of this kind of partnership. Writing not only deepened these students' understandings, it also helped the students relate the mathematics they learned in the classroom to the real world. For example, when students solved the 'City Bus Problem', they were able to analyze the problem in practical terms. They deduced that the answer to the problem did not make sense because in the real world buses are not designed to hold the number of people that the answer specified. This indicated that the students made real-world connections because their metacognitive knowledge aided their understanding of the problem and helped them apply that knowledge to real-world situations. Mayher, Lester, and Pradle (1983) posited that when students make these types of connections, it helps foster an understanding that moves beyond regurgitating facts and definitions, and mimicking procedures illustrated by others. The authors claimed that learning for understanding is a meaning-making process that involves the learner in actively building connections between what is being learned and what is already known.

It became clear that problem-solving activities involving both collaborative conversations and writing activities changed students' attitudes about writing. Students indicated that writing helped them keep track of their thinking and solutions. The think-aloud protocols indicated that students were able to think positively about both writing and problem solving in general. Also, students easily engaged in a discourse that modeled how they would explain a certain problem to a younger student. Given the opportunity, students will construct much of the mathematics they are expected to learn as they attempt to make sense of concepts and ideas either through writing or interactions with their teacher or peers.

Another important insight emerging from this study is that writing strengthened students' individual problem-solving performance because it provi-

ded a context that allowed them to engage in 'self-talk'. This was done in the zone of proximal practice (ZPP). Over the course of the study, students' writing became clearer and more concise regarding strategies and procedures used to solve problems. This 'demands detachment from the actual situation and requires deliberate analytical action on the part of the [student]' (Vygotsky, 1986, p. 182). In the intrapsychological plane, students' individual writing activities were separated from the interpsychological plane (group activities); therefore, according to Vygotsky, the intrapsychological plane was reconstructed through the interpsychological plane. It was in the intrapsychological plane in which students constructed ways to represent their thinking independently in writing. Then the writing became a visual representation of students' individual thought processes.

Wertsch advocated a similar position in a 1979 analysis that focused on the emergence of self-regulative capacities. Wertsch identified situations in the transition from other-regulation to self-regulation in developing a framework to portray points in which problem solving process is carried out on the interpsychological plane as evidenced in the ZPD. In his elaborate explanation, Wertsch made it clear that when individuals carry out tasks unassisted, the 'problem solving activity shifts from the interpsychological to the intrapsychological plane and the transition from other-regulation to self-regulation is completed' (p. 19). Although Wertsch did not call the completed unassisted level a new learning zone, he implied that self-regulation does not happen in the ZPD.

> ... The further we go beyond the transition from interpsychological to intrapsychological functioning in connection with a particular problem-solving task, the less direct will be the connection between the external social interaction involved in other-regulation and the psychological processes involved in self-regulation.þthe social interaction origins of this individual functioning will still be a necessary part of any adequate account (p. 18).

The theoretical notion of the ZPP provides a way for us to understand and conceive of writing as a mediated tool which has a central place in assisting students in the regulation of their individual thought processes. The ZPD supports the development of students' thinking and the ZPP is where their thinking is further developed. It is in this zone (ZPP) in which individual students revised their writing by either expanding, reorganizing, and/or self-scaffolding their mathematical ideas.

CONCLUSIONS AND IMPLICATIONS

An analysis of these findings provides the basis for the following conclusions and implications. This study supports the notion that writing assists in the development of students' individual thought processes and

helps students construct knowledge about mathematical ideas and concepts. This study illustrates a connection between students' oral thought processes within the zone of proximal development ('outside-in'), and written thought processes within the zone of proximal practice ('inside-out').

In attempting to link the ZPD to the ZPP, the intent was to maintain the idea central to Vygotsky's theoretical construct of the ZPD: the premise that concepts, when initially acquired by students, occur externally in a social context with teachers or peers. These concepts or understandings are further developed when students 'elaborate and differentiate' (Bruner and Haste, 1987, p. 17) their thoughts through speaking or writing. Students' writings are expressions of the differentiation they make as a result of engaging in collaborative conversations. It is through conversations with their peers or an adult that allows students to gradually turn interpersonal talk into intrapersonal talk, and helps them organize oral and written thought processes.

As suggested by this study, learning to express mathematical understandings and concepts via writing is not an isolated process. Its expression depends upon being involved in active learning situations, as actualized by the ZPD. The ZPD further serves as a catalyst for independent thinking that emerges within the ZPP. Mathematics need not be learned in rote or rigid ways. Rather students can apply their knowledge to specific tasks or situations that involve active, constructive, and innovative practices, which can be manifested through writing.

Students sometimes struggle to create mathematical understandings of complex problems that apply to real-world situations. Therefore, the mathematics environment should be constructed in ways that encourage students to find their paths to solutions and connect those paths to their ideas. Furthermore, it is important that students experience mathematics as a process of exploration in which their experiences help empower them as learners, and more specifically, as problem solvers. 'Teaching mathematics from a problem-solving perspective entails more than solving non-routine, but often isolated problems ... . It involves the notion that the very essence of studying mathematics is itself an exercise in exploring, conjecturing, examining and testing all aspects of problem-solving' (NCTM, 1991, p. 95).

While it is important for students to solve problems with prescribed steps or procedures, they must also think about what they are doing, how they are doing it, and why they are doing it. Equally important is that students share their work with other students and with their teacher, sharing can help bridge gaps or misunderstandings they may have about the

mathematics they are learning. The social nature of mathematical communication must be an integral and substantial part of the learning process. Dynamic mathematical communication is critical to learning and understanding mathematical concepts and ideas in which thought processes are developed 'outside-in' through mediated practices assisted by others – to thought processes developed 'inside-out' through mediated practices assisted by self.

It is suggested that similar studies be conducted at various grade levels to develop a broader picture of what it means to create new learning zones as proposed by Newman and Holzman (1993). The authors contend that, 'the continuous creation of ZPDs and environments for making ZPDs and thereby learning leading development, is ... the basic Vygotskian model' (pp. 171). Certainly, new knowledge and information that emerge from further studies and from the creation of new learning zones will assist in the improvement of teaching and learning of problem solving and mathematics in general. As implied in this study, the ZPD and the ZPP are powerful learning environments in which to develop students' mathematical thought processes.

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APPENDIX A INTERVIEW PROTOCOL

Problem Comprehension

1. What is this problem about? What can you tell me about it?
2. Is there something that can be eliminated or that is missing?
3. What assumptions do you have or make?

Approaches and Strategies

4. Where could you find the needed information?
5. What steps did you take?
6. How did you organized the information?
7. Did you have a system? A strategy?
8. Would it help to draw a diagram or make a chart?

Relationships

9. What is the relationship of this to that?
10. Is there a pattern?
11. Can you write another problem related to this one?

Flexibility

12. Would another approach work as well or better?
13. Would writing a statement help explain how you obtained your answer?

Communication

14. How would you explain what you know right now?
15. How would you explain this process to a younger child?
16. Could you write an explanation for next year's students about how to do this?
17. Which words are most important? Why?

Examining Solutions and Results

18. Is that the only possible answer?
19. Is the solution reasonable, considering the context?
20. What made you think that was what you should do?
21. Is there a real-life situation where this could be used?
22. What questions does this raise for you?

Mathematical Learning and Self-Assessment

23. What were the mathematical ideas in this problem?
24. How many kinds of mathematics were used in this investigation?
25. What do you need to do next?
26. What have you accomplished?

Stenmark, 1991, pp. 31–32

REFERENCES

Albert, L.: 1995, 'A case study on the complexities of learning to teach problem solving: The effects writing process strategy model on seventh grade students', Mathematics Performance,' Doctoral Dissertation, University of Illinois at Urbana-Champaign, 1995, *Dissertation Abstracts International*, 57, A1529.

Burkhalter, N.: 1995, 'A Vygotsky-based curriculum for teaching persuasive writing in the elementary grades,' *Language Arts* 72, 192–199.

Bruner, J.S.: 1971, *The Relevance of Education*, Norton, New York.

Bruner, J.S.: 1973, *Beyond the Information Given: Studies in the Psychology of Knowing*, Norton, New York.

Bruner, J.S.: 1975, 'The ontogenesis of speech acts,' *Journal of Child Language* 2, 1–21.

Bruner, J.S. and Haste: 1987, 'Introduction', in J. Bruner and H. Haste (eds.), *Making Sense: The Child Construction of the World*, Methuen, New York, pp. 1–25.

Bruner, J.S., Oliver, R. and Greenfield, P.: 1966, *Studies in Cognitive Growth*, Son, New York.

Cole, M.: 1996, *Cultural Psychology: A Once and Future Disciplines*, Harvard University Press, Cambridge, MA.

Creswell, J.W.: 1998, *Qualitative Inquiry and Research Design: Choosing Among Five Traditions*, Sage Publications, Thousand Oaks, CA.

DiCamilla, F.J. and Anton, M.: 1997, 'Repetition in the collaborative discourse of L2learners: A Vygotskian perspective', *The Canadian Modern Language Review* 53, 609–633.

Doolittle, P.E.: 1991, 'Vygotsky and the socialization of literacy', Eric Document ED 377 473.

Elbow, P.: 1981, *Writing with Power*, Oxford University Press, New York.

Emig, J.A.: 1983, *The Web of Meaning*, Boynton/Cook, Upper Montclair, NJ.

Ernest, P.: 1998, *Social Constructivism as a Philosophy of Mathematics*, State University of New York Press, New York.

Everson, B.J.: 1991, 'Vygotsky and the teaching of writing', *The Quarterly of the National Writing Project and the Center for the Study of Writing* 3, 8–11.

Freedman, S.W.: 1995, 'Crossing the bridge to practice', *Written Communication* 22, 74–92.

Graves, D.: 1983, *Writing: Teachers and Children at Work*, Heinemann Educational Books, Portsmouth, NH.

Hoel, T.L.: 1997, 'Voices from the classroom', *Teaching and Teacher Education* 13, 5–16.

John-Steiner, V.: 1985, *Notebooks of the Mind*, Harper and Row, New York.

Lave, J. and Wenger, E.: 1991, *Situated Language: Legitimate Peripheral Participation*, Cambridge University Press, Cambridge.

Lantolf, J.P. and Pavlenko, A.: 1995, 'Sociocultural theory and second language acquisition', *Annual Review of Applied Linguistics* 15, 108–124.

Lemke, J.: 1988, 'Genres, semantics, and classroom education', *Linguistics and Education* 1, 81–99.

Luria, A.R. and Yudovich, F.I.: 1971, *Speech and the Development of Mental Processes in the Child*, Penguin, Baltimore.

Mayher, J.S., Lester, N.B. and Pradle, G.M.: 1983, *Learning to Write: Writing to Learn*, Boynton Cook, Upper Montclair, NJ.

Moffett, J.: 1981, *Active Voices*, Heinemann Educational Books, Portsmouth, NH.

National Council of Teachers of Mathematics: 1991, *Professional Standards for Teaching Mathematics*, National Council of Teachers of Mathematics, Reston, VA.

Newman, F. and Holzman, L.: 1993, *Lev Vygotsky: Revolutionary Scientist*, Routledge, New York.

Pellegrini, A.D.: 1981, 'The development of preschoolers' private speech, *Journal of Pragmatics*, 5, 445–458.

Polya, G.: 1945, *How to Solve It: A New Aspect of Mathematical Method*, Princeton University Press, Princeton, NJ.

Rogoff, B. and Wertsch, J.V.: 1984, 'Children learning in the 'zone of proximal development', in B. Rogoff and J. Wertsch (eds.), *New Directions for Child Development*, Jossey-Bass, San Francisco.

Rogoff, B.: 1990, *Apprenticeship in Thinking: Cognitive Development in Social Context*, Oxford University Press, New York.

Russell, D.R.: 1997, 'Writing and genre in higher education and workplaces: A review of studies that use cultural–historical activity theory', *Mind, Culture, and Activity* 4(4), 224–237.

Scribner, S.: 1997, 'Mind in action, a functional approach to thinking', in M. Cole, Y. Engestrom and O. Vasquez (eds.), *Mind, Culture, and Activity*, Cambridge University Press, New York, pp. 354–368.

Scribner, S. and Cole, M.: 1981: *The Psychology of Literacy*, Harvard University Press, Cambridge, MA.

Smith, F.: 1988, *Joining the Literacy Club*, Heinemann Educational Books, Portsmouth, NH.

Stake, R.E.: 1994, 'Case studies', in N. Denzin and Y. Lincoln (eds.), *Handbook of Qualitative Research*, Sage, Thousand Oaks, CA, pp. 236–247.

Stenmark, J.K.: 1991, *Mathematics Assessment: Myths, Models, Good Questions, and Practical Suggestions*, National Council of Teachers of Mathematics, Reston, VA.

Tudge, J.: 1990, 'Vygotsky, the zone of proximal development, and peer collaborations: Implications for classroom practice', in L. Moll (ed.), *Vygotsky and Education: Instructional Implications and Applications of Sociohistorical Psychology*, Cambridge University Press, New York, pp. 155–172.

Vygotsky, L.S.: 1978, *Mind in Society: The Development of Higher Psychological Processes*, Harvard University, Cambridge, MA.

Vygotsky, L.S: 1981, 'The genesis of higher mental functions', in J. Wertsch (ed.), *The Concept of Activity in Soviet Psychology*, M. E. Sharpe, New York, pp. 144–188.

Vygotsky, L.S.: 1986, *Thought and Language* (A. Kozulin, Trans.), M. I. T. Press, Cambridge, MA (original work published 1934).

Wertsch, J.V.: 1979, 'From social interaction to higher psychological processes: A clarification and application of Vygotsky's theory', *Human Development* 22, 1–22.

Wertsch, J.V.: 1979, 'The regulation of human action and the given-new organization of private speech', in G. Zivin (ed.), *The Development of Self-Regulation Through Private Speech*, Wiley, New York, pp. 79–98.

Wertsch, J.V.: 1985, *Vygotsky and the Social Formation of Mind*, Harvard University Press, Cambridge, MA.

Wertsch, J.V.: 1991, *Voices of the Mind: A Sociocultural Approach to Mediated Action*, Harvard University Press, Cambridge, M.A.

Wertsch, J.V.: 1997, 'Collective memory: A sociohistorical perspective', in M. Cole, Y. Engestrom and O. Vasquez (eds.), *Mind, Culture, and Activity*, Cambridge University Press, New York, pp. 226–232.

Boston College, School of Education, Campion Hall, 217
140 Commonwealth Avenue
Chestnut Hill, Massachusetts 02167
U.S.A.
E-mail: albertli@bc.edu